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A Markov–Bernstein Type Inequality for Algebraic Polynomials in L_p, $0<p<1$*

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We prove a weighted inequality for algebraic polynomials and their derivatives in $L_p[-1,1]$ when $0<p<1$. This inequality plays the same role in the proofs of inverse theorems for algebraic polynomial approximation in L_p as the classical Bernstein inequality does in the case of trigonometric polynomials.

1. STATEMENT OF THE THEOREM

We prove the following weighted inequality for algebraic polynomials and their derivatives in L_p when $0<p<1$:

THEOREM 1. *For every algebraic polynomial P_n of degree not exceeding n*

$$\left\| \frac{P_n^{(r)}\rho_n^r}{\omega(\rho_n)} \right\|_{L_p[-1,1]} \leqslant c\cdot \left\| \frac{P_n}{\omega(\rho_n)} \right\|_{L_p[-1,1]}, \tag{1.1}$$

where $\rho_n(x)=n^{-1}(1-x^2)^{1/2}+n^{-2}$ and the function $\omega\colon R^+\to R^+$ satisfies with some constant $M\geqslant 1$ the condition

$$M^{-1}\omega(t_1)\leqslant\omega(t_2)\leqslant M\cdot\omega(t_1), \qquad \forall t_1\leqslant t_2\leqslant 2t_1. \tag{1.2}$$

The constant c may be represented as $c=c_0^r\cdot r!$, where c_0 depends on M and p, but on nothing else.

The inequality (1.1) plays the same role in the proofs of inverse theorems for algebraic polynomial approximation as the classical Bernstein inequality does for trigonometric polynomials. In the case $p=\infty$ the inequality (1.1) was proved by Yu. Brudnyi [2] and used by A. F. Timan [11] in the proof

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of the inverse theorem for approximation by algebraic polynomials in the uniform metric. In the case $1 \leqslant p < \infty$ the inequality (1.1) was proved by G. K. Lebed [5] and used by the author [9] in the proof of the inverse theorem for algebraic polynomials approximation in L_p.

For $0 < p < 1$ the inequality (1.1) was proved by P. Nevai [7] and V. I. Ivanov [4] in the special case $\omega(t) = t^\alpha$; however, no explicit statement was made on the dependence of the constant c on r. Such dependence plays an important role when the inequality is being used in inverse theorems, which is a peculiarity of the case $0 < p < 1$ (see, e.g. [10], [3]); we recall that the Bernstein inequality for trigonometric polynomials in L_p, $0 < p < 1$, holds with the constant $c = 1$ (V. V. Arestov, [1]). The special case $\omega(t) \equiv 1$ of the inequality (1.1) with the appropriately estimated constant is contained in the paper of G. Tachev [10] and in the paper of Z. Ditzian, D. Jung, and D. Leviatan [3], and is used there in proofs of the inverse theorems for generalized moduli of smoothness. Inverse theorems for the classical modulus of smoothness require inequalities with the special weights $\omega(\rho_n)$ to handle the "end-effect" of algebraic polynomial approximation (see, e. g., [12]); inverse theorems for Lipschitz spaces in L_p and more general Besov spaces B^s_{pq} when $p \neq q$ require the inequality (1.1) with non-power majorant ω (see [8]).

2. Proof of the Theorem

We start with an auxiliary inequality in L_p, which is similar to Markov's inequality in the uniform metric (see, e.g., [6, p. 39]):

LEMMA 1. *Let $0 < p < 1$ and the function ω: $R^+ \to R^+$ satisfies the condition* (1.2) *with some constant $M \geqslant 1$. For every polynomial P_n of degree not exceeding n*

$$\left\| \frac{P_n'}{\omega(\rho_n)} \right\|_{L_p(A_n)} \leqslant c \cdot n^2 \cdot \left\| \frac{P_n}{\omega(\rho_n)} \right\|_{L_p[-1,1]}, \tag{2.1}$$

where $A_n = \{x \in [-1, 1] : (1-x^2)^{1/2} \leqslant n^{-1}\}$, and the constant c depends on p and M, but on nothing else.

Proof. Let $B_n = \{x \in [-1, 1]] : (1-x^2)^{1/2} > n^{-1}\}$, $s = [p^{-1}] + 1$, and $\mu = s - p^{-1}$. We will write $a \prec b$ if there is a constant $c \geqslant 1$ such that $a \leqslant c \cdot b$; unless an explicit remark is made about the constant c, we will assume that c may depend on p and M, but on nothing else. We will also use the notations $\delta(x) = (1-x^2)^{1/2}$ and $\delta_n(x) = \max\{(1-x^2)^{1/2}, n^{-1}\}$.

Let $T_n(\theta) = P_n(\cos\theta)(\sin\theta)^s$, and $x = \cos\theta$. For $x \in B_n$

$$|P_n'(x)| \prec n^{1+1/p} \cdot \delta(x)^{-\mu} \cdot \{n\,|T_n(\theta)| + |T_n'(\theta)|\}. \tag{2.2}$$

Let $T_n^{(0)} = T_n$, and $T_n^{(1)} = T_n'$. We use the representation

$$T_n^{(\xi)}(\theta) = \int_{-\pi}^{\pi} T_n(\theta + t) \cdot K_{n,m;\xi}(t)\,dt, \qquad \theta \in R, \quad \xi = 0, 1, \tag{2.3}$$

where $K_{n,m;}(t)$ are trigonometric polynomials of degree at most $c_p \cdot m \cdot n$, and

$$|K_{n,m;\xi}(t)| \prec n^{1+\xi} \cdot (1 + n\,|t|)^{-m}, \qquad t \in [-\pi, \pi], \quad \xi = 0, 1, \tag{2.4}$$

with the constant independent of t and n (see, e.g., [5, Lemmas 2 and 6], [12, Section 4.7.5]). Applying (2.3), (2.4), and the "different metrics" inequality [4]

$$\|\tau\|_{L_1[-\pi,\pi]} \prec N^{1/p-1}\, \|\tau\|_{L_p[-\pi,\pi]}, \tag{2.5}$$

for trigonometric polynomials τ of degree at most N, in our case $\tau(u) = T_n(\theta + u) \cdot K_{n,m;\xi}(u)$, we obtain the estimate

$$|T_n^{(\xi)}(\theta)| \prec n^{1/p+\xi} \cdot \|T_n(\theta + \cdot)(1 + n\,|\cdot|)^{-m}\|_{L_p[-\pi,\pi]} \tag{2.6}$$

with the constant depending on p and m, but on nothing else.

Let $x \in B_n$, $x = \cos\theta$, and $m \geqslant \log_2 M + 1$. It follows from the estimates (2.2) and (2.6) that

$$|P_n'(x)| \prec n^{2(1+1/p)} \cdot \delta(x)^{-\mu} \cdot \|T_n(\theta + \cdot)(1 + n\,|\cdot|)^{-m}\|_{L_p[-\pi,\pi]}. \tag{2.7}$$

The function $\varphi_n(u) = \omega(\rho_n(u))$ satisfies the condition

$$\varphi_n(\cos(\theta + t)) \leqslant c_0 (1 + n\,|t|)^{\lambda}\, \varphi_n(\cos\theta), \qquad \text{if} \quad |\sin\theta| \geqslant n^{-1}, \tag{2.8}$$

with $c_0 = M^2$, and $\lambda = \log_2 M$. The estimate

$$\|T_n(\theta + \cdot)(1 + n\,|\cdot|)^{-m}\|_{L_p[-\pi,\pi]} \prec \delta(x)^{\mu} \cdot \omega(\rho_n(x)) \cdot \left\| \frac{P_n}{\omega(\rho_n)} \right\|_{L_p[-1,1]} \tag{2.9}$$

follows directly from the definitions, the inequality $|\sin(\theta + t)| \leqslant (1 + n\,|t|) \cdot |\sin\theta|$, and (2.8). The inequalities (2.7) and (2.9) imply that for every $x \in B_n$

$$|P_n'(x)| \prec L \cdot \omega(\rho_n(x)), \tag{2.10}$$

where $L = n^{2(1+1/p)} \cdot \|P_n / \omega(\rho_n)\|_{L_p[-1,1]}$.

The inequality (2.10) can be extended to the whole interval $[-1, 1]$ by using the standard argument (see, e.g., [12, p. 221–222] or [5, Theorem 3]). The required inequality (2.1) follows from (2.10), when $x \in A_n$, and the estimate $mes A_n \leqslant 2n^{-2}$. ▮

We now prove an auxiliary inequality, which is similar to Bernstein's inequality for algebraic polynomials in the uniform metric (see, e.g., [6, p. 40]):

LEMMA 2. *Let $0 < p < 1$, $n \in N$, and the function $\varphi_n\colon R^+ \to R^+$ satisfies the condition* (2.8) *with some constants $c_0 \geqslant 1$ and $\lambda > 0$. For every polynomial P_n of degree not exceeding n*

$$\left\| \frac{P'_n \cdot \delta}{\varphi_n} \right\|_{L_p(B_n)} \leqslant c \cdot n \cdot \left\| \frac{P_n}{\varphi_n} \right\|_{L_p[-1,1]}, \tag{2.11}$$

where $B_n = \{x \in [-1, 1] : (1-x^2)^{1/2} > n^{-1}\}$, $\delta(x) = (1-x^2)^{1/2}$, and the constant $c \geqslant 1$ depends on c_0, λ, and p, but on nothing else.

Proof. Let $s = [p^{-1}] + 1$, $\mu = s - p^{-1}$, and $T_n(\theta)$ is the trigonometric polynomial used in the proof of Lemma 1. For $x = \cos\theta$

$$|P'_n(x) \cdot \delta(x)| \prec |P_n(x)| \cdot \delta(x)^{-1} + |T'_n(\theta)| \cdot |\sin\theta|^{-s}.$$

Let $\bar{B}_n = \{\theta \in [-\pi, \pi] : n^{-1} \leqslant |\sin\theta|\}$. Then

$$\left\| \frac{P'_n \cdot \delta}{\varphi_n} \right\|_{L_p(B_n)} \prec n \cdot \left\| \frac{P_n}{\varphi_n} \right\|_{L_p(B_n)} + \left\| \frac{T'_n(\theta)(\sin\theta)^{-\mu}}{\varphi_n(\cos\theta)} \right\|_{L_p(\bar{B}_n)}, \tag{2.12}$$

Using the inequality (2.8), the estimate $|\sin(\theta + t)| \leqslant (1 + n\,|t|) \cdot |\sin\theta|$ for $\theta \in \bar{B}_n$, and the inequality (2.6) with $\xi = 1$ and $m = [\mu + \lambda] + 3$, we obtain

$$\left\| \frac{T'_n(\theta)(\sin\theta)^{-\mu}}{\varphi_n(\cos\theta)} \right\|_{L_p(\bar{B}_n)} \prec n \cdot \left\| \frac{P_n}{\varphi_n} \right\|_{L_p[-1,1]}. \tag{2.13}$$

The required inequality (2.11) follows from (2.12) and (2.13). ▮

The inequality (1.1) in the case $r = 1$ follows from Lemmas 1 and 2 when $\varphi_n(x) = \omega(\rho_n(x))$. To obtain (1.1) with $r > 1$, we use the following lemma:

LEMMA 3. *Let $0 < p < 1$ and the function $\omega\colon R^+ \to R^+$ satisfies the condition* (1.2) *with some constant $M \geqslant 1$. For every polynomial P_n of degree not exceeding $n \in N$ and $v = 1, ..., n$*

$$\left\| \frac{P'_n \delta_n^{v+1}}{\omega(\rho_n)} \right\|_{L_p[-1,1]} \leqslant c \cdot n \cdot v \cdot \left\| \frac{P_n \delta_n^{v}}{\omega(\rho_n)} \right\|_{L_p[-1,1]}, \tag{2.14}$$

where $\delta_n(x) = \max\{(1-x^2)^{1/2}, n^{-1}\}$, *and the constant* c *depends on* p *and* M, *but on nothing else.*

Proof. It follows from Lemma 1 that

$$\left\| \frac{P'_n \delta_n^{\nu+1}}{\omega(\rho_n)} \right\|_{L_p(A_n)} \prec n \cdot \left\| \frac{P_n \delta_n^{\nu}}{\omega(\rho_n)} \right\|_{L_p[-1,1]}, \tag{2.15}$$

since $\delta_n(x) = n^{-1}$ for every $x \in A_n$ and $n^{-1} \leqslant \delta_n(x)$ for $x \in [-1, 1]$.

It follows from Lemma 2 that

$$\left\| \frac{P'_n \delta_n^{\nu+1}}{\omega(\rho_n)} \right\|_{L_p(B_n)} \prec n \cdot \left\| \frac{P_n \delta_n^{\nu}}{\omega(\rho_n)} \right\|_{L_p[-1,1]}, \tag{2.16}$$

Indeed, when ν is even, we apply Lemma 2 to the polynomial $P_n \delta^\nu$ of degree not exceeding $2n$ and the weight function $\varphi_n(x) = \omega(\rho_n(x))$, and use the estimate $|\delta'(x)| \leqslant n$ when $x \in B_n$. When ν is odd, Lemma 2 is applied to the polynomial $P_n \delta^{\nu+1}$ and the weight function $\varphi_n(x) = \delta(x) \cdot \omega(\rho_n(x))$. The function $\delta(x)$ is finally replaced by $\delta_n(x)$, since $\delta_n(x) = \delta(x)$ for every $x \in B_n$, and $\delta(x) \leqslant \delta_n(x)$ for $x \in [-1, 1]$.

The inequality (2.14) follows from (2.15) and (2.16). ∎

Proof of Theorem 1. The inequality (1.1) follows by induction from Lemma 3. ∎

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